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ANALYTICAL MODELS FOR THE PREDICTION OF CREEP DEFORMATIONS IN HOLLOW, FINITE, RIGHT-CIRCULAR CYLINDERS

by

O. E. Widera and R. W. Weeks

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Materials Science Division

January 1970

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ABSTRACT

The analytical prediction of primary and secondary creep deformations in hollow, finite, right-circular cylinders loaded by axial tension or by compression, and subjected to internal and external pressure, and to arbitrary temperature gradients is discussed.

I. INTRODUCTION

An accurate prediction of fuel-element deformations depends in large measure on acquiring a basic knowledge of the creep properties of fuel and cladding materials in a reactor environment. Since most existing creep data are from out-of-pile experiments, the determination of the fission-induced enhancement of creep rates in-pile is of special interest. Although apparently significant only at relatively low temperatures, this effect may prove to be important because the cooler peripheral areas of the fuel elements carry the greatest load.

Experiments are presently under way on fuel-element materials (for example, Refs. 1 and 2) both to determine the effect of neutron flux on creep rates and to obtain basic creep data of a more general nature. The creep specimens used in many of these studies are hollow, right-circular cylinders loaded by axial tension or compression. In order to isolate the in-pile creep effect, the analytical predictions of creep deformations based on out-of-pile data will be compared with the test results for low-temperature, in-pile creep.

Analytical models, useful both for analysis of creep-test results with hollow cylinders and, subsequently, for predicting fuel-element creep behavior, are developed in this report. In practice, the models will be used first to determine the correct form of the creep law during out-of-pile tests, and then to predict creep behavior under reactor operating conditions.

A. Background

Many studies concerning the development of analytical models for creep deformations have been published. Bibliographies of the macroscopic investigations may be found in Oding[3] and in Finnie,[4] and, on the microscopic level, Garofalo[5] and Dorn[6] both give good summaries. In 1959, Mendelson et al.[7] outlined a successive approximation method for the treatment of creep in plane stress problems. Their general approach has served as the basis for considerable subsequent work in the analysis of creep deformations and will be employed in this investigation. Included among other work relevant to present needs is that of Wilson and Davis,[8] who extended the Mendelson technique of successive approximations to include the generalized plane strain analysis of creep in a closed cylinder resulting from internal pressure.

In a different approach, Smith[9] analyzed primary creep in thick tubes under a pressure loading by finite-difference methods. Yalch and McConnelee[10] also used a finite-difference technique to solve the differential equations for the creep of composite tubes under radial and axial pressure loads with radial temperature variation. In this latter work, the differential equations were solved numerically by a direct computer approach.

Still another method of attack was given by Besseling,[11] who presented a numerical scheme for axially symmetric creep problems by using an extremum principle for the rate of deformation. More recently, Greenbaum and Rubinstein[12] developed a finite-element routine for the analysis of creep in axisymmetric bodies under axisymmetric-loading conditions.

Finally, Blackburn[13] investigated creep of thin tubes, both singly and in a concentric arrangement, under conditions of internal pressure and cyclic thermal loading. Although the analysis is limited, due to neglect of elastic effects, axial strain, and shear stresses, Blackburn does attempt to apply his results to the deformation of reactor fuel elements.

Two approaches are employed in the present report to model the creep deformations of hollow-cylindrical specimens under the given multiaxial stress conditions. The problem is treated first by a generalized plane strain approach and then, more generally, by a three-dimensional variational technique. Both methods will employ the technique of successive approximations; however, the variational solution can account for end effects and axial temperature gradients, whereas the plane strain approach cannot. Each solution allows for complete flexibility in the choice of creep law and loading path.

A two-shell model will also be developed to examine the effects of different creep rates in adjoining concentric cylinders. With internal pressure specified as a function of time, the two-shell model may serve as a crude approximation to fuel-element behavior.

B. Creep Laws

Creep is the irreversible deformation with time of a body subjected to a sustained lo . For metals, this time-dependent plastic deformation usually becomes important only at elevated temperatures, but, as noted, an enhancement of creep rates may occur in-pile that is significant at low temperatures.

Each material, in principle, undergoes a characteristic creep deformation (strain versus time, Fig. 1) for given mechanical and thermal loading conditions. The terminology primary, secondary, and tertiary stages of creep correspond, respectively, to periods of decreasing, constant, and increasing strain rate. The tertiary stage ends with creep rupture. For purposes of modeling normal fuel-element behavior, the primary and secondary stages of creep are of most interest since the onset of tertiary creep, at least in cladding materials, must be avoided.

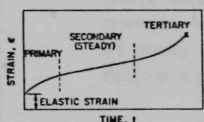

Fig. 1

Typical Creep Curve for Metals. The three stages of creep in a uniaxial test are shown.

Certain aspects of the physical microprocesses that comprise macroscopically observed creep can be explained on the basis of atomic diffusion, dislocation motion, and intergranular flow. A quantitatively complete microstructural theory, howev , is not yet available. Creep "laws" (strain-stress-time-temperature relat ns), therefore, are usually empirical formulations designed tc fit the test data of macroscopic creep. Since many types of creep laws have been proposed, one should note that the analysis developed in this report is sufficiently flexible to handle an arbitrary creep-law formulation.

Two common approaches to creep laws seem to be most favored. The first, rheological modeling, consists of building linear models of materials by mathematically forming various series and parallel combinations of springs and dashpots. By employing more combinations, one has available more adjustable parameters with which to fit experimental data. Such models are especially desirable from an analytical point of view because of their linearity, and the theory of linear viscoelasticity is based on these models (see, for example, Ref. 14). However, if nonlinearity must be included, Coulomb friction elements may be added to the network, or the springs and dashpots can be endowed with nonlinear properties.

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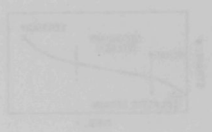

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A creep law with time dependence that results from a spring and dashpot in parallel (Kelvin model), coupled with another spring and dashpot in series (Maxwell model), can be written as follows:

$$\epsilon^c = A_0\sigma^n e^{-Q/RT}(1 + Bt - Ce^{-Dt}), \tag{1}$$

where ϵ^c is the creep strain, σ is the applied stress, t is the time, Q is an average activation energy for creep mechanisms, R is the universal gas constant, T is the absolute temperature, and A_0, n, B, C, and D are experimental constants. This equation reflects the belief that creep is an Arhennius-type thermally activated process that involves nonlinear viscous flow.

Although rheological modeling theoretically offers great flexibility, quite often, as in the case of fuel-element materials, the available experimental data are very incomplete and at times inconsistent. The possibility of many adjustable parameters is of no advantage in such cases, and the following traditional, and very simple, formulation is often adequate:

$$\epsilon^c = A_0\sigma^n e^{-Q/RT} t^m, \tag{2}$$

where m is an experimentally determined constant, and the other symbols have the same meaning as previously defined.

When the effect of the fission process on the creep rate is known, Eqs. 1 and 2 can be modifed in a suitable manner. Cornfield et al.[1] suggest either a simple additive or multiplicative term proportional to the fission rate. In a recent study of Zircaloy, Nichols[2] prefers an additive formulation.

When differentiated with respect to time, holding stress and temperature constant during each successive time step in a quasi-static approach, Eq. 2 leads to a time-hardening law:

$$\dot{\epsilon}^c = A_0 m\sigma^n e^{-Q/RT} t^{m-1}$$

or

$$\Delta\epsilon^c = A_0 m\sigma^n e^{-Q/RT} t^{m-1}\,\Delta t. \tag{3}$$

Solving Eq. 2 for time t and substituting into Eq. 3 yields a strain-hardening law:

$$\dot{\epsilon}^c = m\epsilon^{(m-1)/m}[A_0\sigma^n e^{-Q/RT}]^{1/m}$$

or

$$\Delta\epsilon^c = m\epsilon^{(m-1)/m}[A_0\sigma^n e^{-Q/RT}]^{1/m}\,\Delta t. \tag{4}$$

For many materials, this law seems to work reasonably well[3,4] during constant or increasing loading; however, since a reactor may be run for intervals of time (for example, overnight) under a decreased load, some annealing of the strain-hardening and the radiation damage will occur. To account for this, Gittus[15] has proposed that when the stress drops below some given level, say at time t_0, then the strain ϵ in Eq. 4 should be replaced by an "effective strain" given by

$$\epsilon^* = \epsilon e^{-t'/\tau}, \tag{5}$$

where τ is an experimentally measured time constant and

$$t' = t_0 e^{-Q/RT}. \tag{6}$$

This simple expedient embodies the idea that the annealing will be essentially a relaxation phenomena with an Arhennius time dependence. For definiteness, the creep law represented by Eqs. 4-6 will be assumed in subsequent sections of this report.

II. MULTIAXIAL STRESS-STRAIN RELATIONS

Creep laws generally represent the equation of state for a uniaxial state of stress. The constitutive equations for creep under multiaxial stress conditions can be derived from the following conditions:

1. The constitutive equations for a uniaxial state of stress should result when a multiaxial state of stress degenerates into a uniaxial state.

2. The equations should express volume constancy for the creep deformation. This results from the plastic nature of creep.

3. A superimposed hydrostatic stress should not give rise to any change in the creep rate. (This assumption has been the subject of some controversy.[16])

4. The principal directions of the rates of strain and stress tensors should coincide for an isotropic medium.

The flow or incremental theory of plasticity states that the increments of plastic strain are related to the final state of stress, the plastic strain, and the stress increment. In general these relations are not integrable, and the integral depends on the loading path:

$$\epsilon_{ij}^{p} = \int_{\text{path}} \Delta\epsilon_{ij}^{p} \tag{7}$$

and

$$\Delta \epsilon_{ij}^{p} = \Delta \epsilon_{ij}^{p}(\sigma_{kl}, \epsilon_{kl}^{p}, \Delta \sigma_{kl}), \tag{8}$$

where $\Delta \epsilon_{ij}^{p}$ are the infinitesimal plastic strain increments, ϵ_{ij}^{p} are the plastic strain components, σ_{kl} is the stress tensor, and the subscripts i, j, k, l = 1, 2, 3. Equation 7 insures history dependence.

Assuming that the von Mises yield criterion holds, the associated stress-strain relations can be written as

$$\Delta \epsilon_{ij}^{p} = \frac{3}{2} \frac{\Delta \epsilon_{p}}{\sigma_{e}} S_{ij}, \tag{9}$$

where

$$S_{ij} = \sigma_{ij} - \sigma \delta_{ij}, \tag{10a}$$

$$\sigma = 1/3 \; \sigma_{kk}, \tag{10b}$$

$$\sigma_{e} = (3/2 \; S_{ij} S_{ij})^{1/2}, \tag{10c}$$

and

$$\Delta \epsilon_{p} = (2/3 \; \Delta \epsilon_{ij}^{p} \Delta \epsilon_{ij}^{p})^{1/2}. \tag{10d}$$

Here, S_{ij} are the deviatoric stress components, σ_e is the equivalent stress, $\Delta \epsilon_p$ is the equivalent plastic-strain increment, and δ_{ij} is the Kronecker delta. Repeated indices imply the use of the summation convention. Equations 9 and 10 are generally known as the Prandtl-Reuss relations.

Although the physical processes involved in creep may be different from those occurring in plastic flow, we will assume that the relations relating stress to creep strain are given by the Prandtl-Reuss equations. Condition 1 is then satisfied if the equivalent creep-strain increment is related to the equivalent stress in the same way that one-dimensional creep is related to the tension or compression test. The other conditions are satisfied by the Prandtl-Reuss equations.

III. GENERALIZED PLANE STRAIN SOLUTIONS

A. Method of Analysis

With reference to a cylindrical coordinate system (r, θ, z), the differential equation of equilibrium for axisymmetric problems in the state of generalized plane strain is given by

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{11}$$

The strain-displacement relations are

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \text{constant}, \tag{12}$$

where u is the radial displacement.

The solution of creep problems can, in general, be obtained by using an incremental approach. One starts with a given increment of time and solves the problem by successive approximations. The next increment of time is then chosen, and the solution proceeds in the same manner. Let ϵ_{ij}^c be the components of the total creep strain at time t and $\Delta\epsilon_{ij}^c$ the additional increment of creep strain during time interval Δt. For small strains, we then have the following stress-strain relations:

$$\sigma_r = 2G\left(\epsilon_r + \frac{\nu\epsilon}{1 - 2\nu} - \epsilon_r^c - \Delta\epsilon_r^c - \frac{1 + \nu}{1 - 2\nu}\alpha T\right);$$

$$\sigma_\theta = 2G\left(\epsilon_\theta + \frac{\nu\epsilon}{1 - 2\nu} - \epsilon_\theta^c - \Delta\epsilon_\theta^c - \frac{1 + \nu}{1 - 2\nu}\alpha T\right);$$

$$\sigma_z = 2G\left(\epsilon_z + \frac{\nu\epsilon}{1 - 2\nu} - \epsilon_z^c - \Delta\epsilon_z^c - \frac{1 + \nu}{1 - 2\nu}\alpha T\right), \tag{13}$$

where G is the shear modulus, ν is the Poisson ratio, α is the coefficient of linear expansion, and

$$\epsilon = \epsilon_r + \epsilon_\theta + \epsilon_z. \tag{14}$$

On substituting Eqs. 12 and 13 into the equilibrium equations, we obtain the following equation for the radial displacement:

$$\left(\frac{1 - \nu}{1 - 2\nu}\right)\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) = \frac{d}{dr}(\epsilon_r^c + \Delta\epsilon_r^c)$$

$$- \frac{(\epsilon_\theta^c + \Delta\epsilon_\theta^c) - (\epsilon_r^c + \Delta\epsilon_r^c)}{r} + \left(\frac{1 + \nu}{1 - 2\nu}\right)\alpha\frac{dT}{dr}. \tag{15}$$

Integrating Eq. 15 twice with respect to r yields

$$u = \frac{d_1}{r} + d_2 r + \left(\frac{1 - 2\nu}{1 - \nu}\right)\frac{1}{r}\left[\int_a^r (\epsilon_r^c + \Delta\epsilon_r^c)\, r_1\, dr_1\right.$$

$$\left. - \int_a^r r_1 \int_a^{r_1} \frac{(\epsilon_\theta^c + \Delta\epsilon_\theta^c) - (\epsilon_r^c + \Delta\epsilon_r^c)}{r_2}\, dr_2 dr_1\right] + \left(\frac{1 + \nu}{1 - \nu}\right)\frac{\alpha}{r}\int_a^r T r_1\, dr_1. \tag{16}$$

where d_1 and d_2 are constants of integration. This expression for u, upon integration by parts, can be rewritten as

$$u = \frac{d_1}{r} + d_2 r + \left(\frac{1 - 2\nu}{1 - \nu}\right)\frac{r}{4G}[A(r) - B(r)] + \left(\frac{1 + \nu}{1 - \nu}\right)\alpha r \int_a^r T r_1 \, dr_1, \tag{17}$$

where

$$A(r) = \frac{2G}{r^2}\int_a^r (\epsilon_\theta^c + \Delta\epsilon_\theta^c + \epsilon_r^c + \Delta\epsilon_r^c)\, r_1 \, dr_1$$

and

$$B(r) = 2G\int_a^r (\epsilon_\theta^c + \Delta\epsilon_\theta^c - \epsilon_r^c - \Delta\epsilon_r^c)\frac{1}{r_1}\, dr_1. \tag{18}$$

In terms of Eq. 18, the stresses can now be expressed as follows:

$$\sigma_r = \frac{2G}{1 - 2\nu}\left[d_2 - (1 - 2\nu)\frac{d_1}{r^2} + \nu\epsilon_z\right] - \frac{2G(1 + \nu)\,\alpha}{1 - \nu}\,\theta(r)$$

$$- \frac{1}{1 - 2\nu}[(1 - 2\nu)\,A + B], \tag{19}$$

$$\sigma_\theta = \frac{2G}{1 - 2\nu}\left[d_2 + (1 - 2\nu)\frac{d_1}{r^2} + \nu\epsilon_z\right] - \frac{2G(1 + \nu)\,\alpha}{1 - \nu}\,\theta(r)$$

$$+ \frac{1}{1 - 2\nu}[(1 - 2\nu)\,A - B] - 2G\left[\epsilon_\theta^c + \Delta\epsilon_\theta^c - \left(\frac{\nu}{1 - \nu}\right)(\epsilon_r^c + \Delta\epsilon_r^c)\right], \tag{20}$$

and

$$\sigma_z = 2G\left\{\frac{\nu}{1 - 2\nu}\left[2d_2 - \frac{1}{2G}\left(\frac{1 - 2\nu}{1 - \nu}\right)B\right] - \left(\frac{1 + \nu}{1 - \nu}\right)\alpha T\right.$$

$$\left. + \left(\frac{1 - \nu}{1 - 2\nu}\right)\epsilon_z - \epsilon_z^c - \Delta\epsilon_z^c + \left(\frac{\nu}{1 - \nu}\right)(\epsilon_r^c + \Delta\epsilon_r^c)\right\}, \tag{21}$$

where

$$\theta(r) = \frac{1}{r^2}\int_a^r T r_1 \, dr_1. \tag{22}$$

After d_1, d_2, and ϵ_z have been determined with reference to a particular problem (see next two sections), the solution technique proceeds as follows:

1. Assume $\Delta\epsilon_r^c = \Delta\epsilon_\theta^c = \Delta\epsilon_\theta^c = 0$ at the start.
2. Calculate the elastic stress distribution.
3. Compute σ_e from the stresses.
4. Compute $\Delta\epsilon_c$ from the creep law.
5. Compute $\Delta\epsilon_c$ from the strain increments.
6. Average $\Delta\epsilon_c$ from steps 4 and 5.
7. Calculate $\Delta\epsilon_r^c$, $\Delta\epsilon_\theta^c$, and $\Delta\epsilon_z^c$ from the Prandtl-Reuss relations.
8. Calculate the new stress distribution.
9. Repeat steps 3 through 8 until satisfactory convergence is obtained.
10. At the start of the second time increment, the total creep strains are set equal to the creep-strain increments determined at the end of the first time increment. The procedure of the previous steps is then repeated for this and any other succeeding time interval.

B. Creep of a Finite, Hollow Cylinder

Consider the hollow cylinder in Fig. 2 with internal radius a, external radius b, and length L, subjected to a compressive axial load p and to both internal and external pressure. The boundary conditions can be expressed as

$$\sigma_r = -p_i \qquad (r = a)$$

and

$$\sigma_r = -p_o \qquad (r = b). \tag{23}$$

The condition of equilibrium in the axial direction yields the additional relation

$$\int_a^b \sigma_z r\, dr = -\frac{p}{2\pi} + \frac{1}{2}(p_i a^2 - p_o b^2). \tag{24}$$

For definiteness, let us assume a radial temperature distribution that corresponds to the case of uniform heat generation with no heat transfer at the inner surface.

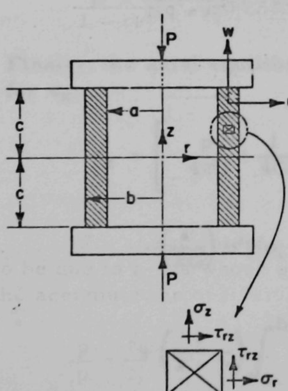

Fig. 2. Coordinate System for a Hollow Cylinder under Compressive Load

Then, as shown in the next section,

$$T(r) = \frac{q}{2k}\left[\frac{b^2 - r^2}{2} + a^2 \ln (r/b)\right] + T_b, \tag{25}$$

where q is the rate of heat generation, k is the thermal conductivity, and T_b is the temperature at r = b. Thus,

$$\theta(r) = \frac{1}{r^2}\left\{\left[\frac{q}{2k}\left(\frac{b^2}{2} - a^2 \ln b\right) + T_b\right] \int_a^r r_1 \, dr_1\right.$$

$$\left. - \frac{q}{4k}\int_a^r r_1^3 \, dr_1 + \frac{qa^2}{2k}\int_a^r r_1 \ln r_1 \, dr_1\right\}. \tag{26}$$

Application of boundary conditions of Eq. 23 now yields

$$d_1 = \left(\frac{a^2b^2}{b^2 - a^2}\right)\frac{1}{2G}[(p_i - p_o) + D]$$

and

$$d_2 = -\left(\frac{1 - 2\nu}{2G}\right) p_i + \left(\frac{1 - 2\nu}{2G}\right)\left(\frac{b^2}{b^2 - a^2}\right)[p_i - p_o) + D] - \nu\epsilon_z, \tag{27}$$

where

$$D = \frac{1}{2(1 - \nu)}[(1 - 2\nu) A(b) + B(b)] + \frac{2G(1 + \nu)\alpha}{1 - \nu}\theta(b). \tag{28}$$

Finally, the axial equilibrium condition yields the following expression for ϵ_z:

$$\epsilon_z = \left[-\frac{P}{4\pi G} + \frac{1}{4G}(p_i a^2 - p_o b^2) + \frac{\nu}{2G}(b^2 - a^2) p_i\right.$$

$$-\left(\frac{\nu}{2G}\right) b^2[(p_i - p_o) + D] - \left(\frac{\nu}{1 - 2\nu}\right)\int_a^b (\epsilon_r^c + \Delta\epsilon_r^c) \, r \, dr$$

$$+\left(\frac{\nu}{1 - \nu}\right)\int_a^b r \int_a^r \frac{1}{r_1}(\epsilon_\theta^c + \Delta\epsilon_\theta^c - \epsilon_r^c - \Delta\epsilon_r^c) \, dr_1 dr$$

$$\left. +\left(\frac{1 + \nu}{1 - \nu}\right)\alpha b^2\theta(b) + \int_a^b (\epsilon_z^c + \Delta\epsilon_z^c) \, r \, dr\right] \frac{2}{(1 + \nu)(b^2 - a^2)}. \tag{29}$$

This completes the formulation of the problem.

C. Creep of Two Concentric Closed Cylinders

Consider the problem of two concentric closed cylinders of finite length under the action of an internal pressure p_1, an external pressure p_2, and an axial force p. The particular application of interest is a nuclear reactor fuel element with a hollow cylinder of fissionable material surrounded by a cylinder of nonfissionable material (cladding). Let a be the inside radius, b the interface radius, and c the outside radius of the composite cylinder. If we assume a frictionless interface, the boundary conditions for this problem can be stated as follows:

$$\sigma_{r_1} = -p_1 \qquad (r = a)$$

and

$$\sigma_{r_2} = -p_2 \qquad (r = c), \tag{30}$$

where the subscripts 1 and 2 refer to the inner and outer cylinders, respectively. In addition, we have the following continuity conditions:

$$\sigma_{r_1} = \sigma_{r_2}; \quad u_1 = u_2 \qquad (r = b). \tag{31}$$

If we assume that the fuel cylinder is somewhat shorter than the closed cladding tube, the values of ϵ_z can be determined from the following axial equilibrium conditions:

$$\int_a^b \sigma_{z1} r\, dr = -\frac{p_1}{2}(b^2 - a^2);$$

$$\int_b^c \sigma_{z2} r\, dr = \frac{p}{2\pi} + \frac{1}{2}(p_1 b^2 - p_2 c^2). \tag{32}$$

In terms of Eq. 19, the boundary conditions in Eq. 21 become

$$-p_1 = \frac{2G_1}{1 - 2\nu_1}\left[d_{21} - (1 - 2\nu_1)\frac{d_{11}}{a^2} + \nu_1 \epsilon_{z1}\right];$$

$$-p_2 = \frac{2G_2}{1 - 2\nu_2}\left[d_{22} - (1 - 2\nu_2)\frac{d_{12}}{c^2} + \nu_2 \epsilon_{z2}\right]$$

$$- \frac{2G_2(1 + \nu_2)\alpha_2}{1 - \nu_2}\theta_2(c) - \frac{1}{1 - 2\nu_2}[(1 - 2\nu_2)A_2(c) + B_2(c)]. \tag{33}$$

The continuity conditions (Eq. 31) yield the following relations:

$$\frac{2G_1}{1-2\nu_1}\left[d_{21} - (1-2\nu_1)\frac{d_{11}}{b^2} + \nu_1 \epsilon_{z1}\right] - \frac{2G_1(1+\nu_1)\,\alpha_1}{1-\nu_1}\,\theta_1(b)$$

$$-\frac{1}{1-2\nu_1}[(1-2\nu_1)\,A_1(b) + B_1(b)] = \frac{2G_2}{1-2\nu_2}\left[d_{22} - (1-2\nu_2)\frac{d_{12}}{b^2} + \nu_2 \epsilon_{z2}\right]$$

and

$$\frac{d_{11}}{b} + d_{21}b + \left(\frac{1-2\nu_1}{1-\nu_1}\right)\frac{b}{4G_1}[A_1(b) - B_1(b)] + \left(\frac{1+\nu_1}{1-\nu_1}\right)\alpha_1 b\theta_1(b) = \frac{d_{12}}{b} + d_{22}b. \quad (34)$$

The equilibrium equations yield the additional two relations needed to determine the values of d and ϵ_z:

$$\frac{G_1}{1-2\nu_1}(b^2-a^2)[2d_{21}\nu_1 + (1-\nu_1)\,\epsilon_{z1}] + 2G_1\int_a^b\left[-\frac{\nu_1}{2G_1(1-\nu_1)}B_1\right.$$

$$\left. -\left(\frac{1+\nu_1}{1-\nu_1}\right)\alpha_1 T_1 - \epsilon_{z1}^c - \Delta\epsilon_{z1}^c + \frac{\nu_1}{1-\nu_1}(\epsilon_{r1}^c + \Delta\epsilon_{r1}^c)\right] r\,dr$$

$$= -\frac{p_1}{2}(b^2-a^2) \quad (35)$$

and

$$\frac{G_2}{1-2\nu_2}(c^2-b^2)[2d_{22}\nu_2 + (1-\nu_2)\,\epsilon_{z2}] + 2G_2\int_b^c\left[-\frac{\nu_2}{2G_2(1-\nu_2)}B_2\right.$$

$$\left. -\left(\frac{1+\nu_2}{1-\nu_2}\right)\alpha_2 T_2 - \epsilon_{z2}^c - \Delta\epsilon_{z2}^c + \left(\frac{\nu_2}{1-\nu_2}\right)(\epsilon_{r2}^c + \Delta\epsilon_{r2}^c)\right] r\,dr$$

$$= \frac{P}{2\pi} + \frac{1}{2}(p_1 b^2 - p_2 c^2). \quad (36)$$

For the problem being considered, the inside pressure p_1 is assumed to be due to a gas whose density is increasing at a constant rate because of the accumulation of fission products. The perfect gas law

$$\frac{\dot{p}}{p} = \frac{\dot{N}}{N} + \frac{\dot{T}}{T} \quad (37)$$

then yields

$$\dot{p} = \frac{p_0}{N_0 T_0} (T\dot{N} + N\dot{T}) \tag{38}$$

or

$$\dot{p} = \frac{p_0}{T_0}[\dot{T} + w(T + t\dot{T})], \tag{39}$$

where T and N are the temperature and density of the gas, respectively, w is the relative rate of increase of density, and a dot above a quantity represents differentiation with respect to time. If the temperature remains constant,

$$p = \frac{p_0}{N_0} N. \tag{40}$$

The pressure on the outer cylinder due to the coolant is assumed constant.

The inner cylinder can be considered to be subjected to a temperature gradient that results from the heat generated by nuclear fission. Assuming a uniform heat source q, the solution of the Fourier heat-conduction equation

$$k_1 \nabla^2 T = -q \tag{41}$$

is given by

$$T_1 = \frac{-qr^2}{4k_1} + c_1 \ln r + c_2, \tag{42}$$

where k_1 is the thermal conductivity of the fuel. When Eq. 41 is used, we assume that all loading (thermal as well as mechanical) takes place in a quasi-static manner.

Application of the boundary conditions

$$\frac{dT}{dr} = 0 \quad (r = a);$$

$$T = T_b \quad (r = b) \tag{43}$$

to Eq. 42 yields

$$T_1 = T_b + \frac{q}{4k_1}[b^2 - r^2 + 2a^2 \ln (r/b)]. \tag{44}$$

The outer cylinder contains no heat source, and the temperature distribution, subject to the boundary conditions

$$T = T_b \qquad (r = b)$$

and

$$T = T_c \qquad (r = c), \tag{45}$$

is given by

$$T_2 = \frac{T_c - T_b}{\ln (c/b)} \ln \frac{r}{b} + T_b. \tag{46}$$

The relation between q_0 and the interface temperature T_b is obtained by equating the fluxes per unit length at $r = b$, which yields

$$T_b = T_c + \frac{q}{2k_2}(b^2 - a^2) \ln (c/b), \tag{47}$$

where k_2 is the thermal conductivity of the cladding.

The equations necessary for the plane strain analysis of the creep deformation of two concentric cylinders are now complete. In order to account for axial temperature variations and general end conditions, a more general three-dimensional approach was also developed.

IV. VARIATIONAL FORMULATION

A. Method of Analysis

For small strains, the strain tensor can be expressed as a sum of elastic, thermal, and creep strains:

$$\epsilon_{ij} = \epsilon_{ij}^E + \epsilon_{ij}^T + \epsilon_{ij}^c + \Delta\epsilon_{ij}^c, \tag{48}$$

where the superscripts E, T, and c refer, respectively, to the elastic, thermal, and creep parts of the strain. Upon removal of the load, the elastic strains can be recovered; therefore, the total strain can be expressed as

$$\epsilon_{ij} = \epsilon_{ij}^E + \eta_{ij}. \tag{49}$$

The term η_{ij} denotes the initial strain, which is given by

$$\eta_{ij} = \epsilon_{ij}^{T} + \epsilon_{ij}^{c} + \Delta\epsilon_{ij}^{c}. \tag{50}$$

One of the most widely used variational principles of classical elasticity theory is the theorem of Minimum Potential Energy:

> *"Of all displacements satisfying the given boundary conditions, those which satisfy the equilibrium equations make the potential energy an absolute minimum."*

The potential energy may be defined as

$$V(u_i) = \int_V \epsilon \, dv - \int_V f_i u_i \, dv - \int_{S_\sigma} t_i u_i \, ds, \tag{51}$$

where

ϵ = strain-energy density,

V = volume,

S_σ = part of the boundary where stresses are specified,

u_i = displacement components,

f_i = components of body force per unit volume,

and

t_i = components of stress vector specified on the boundary.

In linear elasticity theory, the strain-energy density is expressed as

$$\epsilon = \tfrac{1}{2}\sigma_{ij}\epsilon_{ij}. \tag{52}$$

The relations between the components of the strain tensor and the displacement vector are

$$2\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}. \tag{53}$$

The u_i displacements must belong to a class of admissible functions, that is, u_i must satisfy the displacement boundary conditions and must have as many continuous derivatives as required in the solution of the problem.

The theorem of Minimum Potential Energy can be generalized to include creep problems if the strain-energy density is replaced by a modified strain-energy density function $\bar{\varepsilon}$ defined as follows:

$$\bar{\varepsilon} = \tfrac{1}{2}\sigma_{ij}(\epsilon_{ij} - \eta_{ij}). \tag{54}$$

Since the stress tensor is related to the strain tensor by

$$\sigma_{ij} = E_{ijkl}\epsilon^{E}_{kl}$$

$$= E_{ijkl}(\epsilon_{kl} - \eta_{kl}), \tag{55}$$

we can express the modified strain-energy density function in matrix form as

$$\bar{\varepsilon} = \tfrac{1}{2}\{\epsilon\}^T[E]\{\epsilon\} - \{\epsilon\}^T[E]\{\eta\} + \tfrac{1}{2}\{\eta\}^T[E]\{\eta\}, \tag{56}$$

where $[E]$ is the matrix of the elastic moduli and

$$\{\varepsilon\}^T = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{31}, 2\epsilon_{12}\}. \tag{57}$$

The superscript T denotes the matrix transpose. The potential energy is now given by

$$V = \tfrac{1}{2}\int_v \{\epsilon\}^T[E]\{\epsilon\}\,dv - \int \{\epsilon\}^T[E]\{\eta\}\,dv$$

$$+ \tfrac{1}{2}\int_v \{\eta\}^T[E]\{\eta\}\,dv - \int_v \{u\}^T\{f\}\,dv + \int_S \{u\}^T\{t\}\,ds. \tag{58}$$

One of the advantages of stating the problem in a variational form is that a solution is possible with the aid of the Rayleigh-Ritz method, without recourse to the differential equations. The idea of this method is to extremize the functional on a finite-dimensional subspace of admissible functions. For the approximate solution to converge to the true solution, the subspace should contain a set of functions that are relatively complete in the space. The set of functions, which extends over the entire body, is chosen to satisfy the geometric boundary conditions. The convergence can be studied by increasing the dimension of the subspace of admissible functions.

Following the idea of the Rayleigh-Ritz method, we choose the displacements as a linear combination of functions:

$$\underset{3\times 1}{\{u\}} = \underset{3\times n}{[\phi(x_i)]}\{a\}. \tag{59}$$

The strains can be expressed in terms of the a matrix as

$$\{\epsilon\} = [B(x_i)]\{a\}. \tag{60}$$

Substituting this expression into the energy functional yields

$$V = \{a\}^T[k]\{a\} + \frac{1}{2}\int_V \{\eta\}^T[E]\{\eta\}\, dv - \{a\}^T\{Q\} - \{a\}^T\{P\}, \tag{61}$$

where

$$[k] = \int_V [B]^T[E][B]\, dv,$$

$$\{Q\} = \int_V [\phi]^T\{f\}\, dv + \int_S [\phi]^T\{t\}\, ds,$$

and

$$\{P\} = \int_V [B]^T[E]\{\eta\}\, dv. \tag{62}$$

Note that the initial strain in Eq. 61 functions as an additional external load and may be treated as such.

The condition that the energy be an extremum:

$$\delta V = 0, \tag{63}$$

is given by

$$\frac{\partial V}{\partial a_i} = 0 \quad (i = 1, 2, \ldots, n). \tag{64}$$

This condition yields the relations

$$[k]\{a\} = \{Q\} + \{P\}, \tag{65}$$

from which the values of a_j can be determined.

The stresses are determined from the relation

$$\{\sigma\} = [E][B]\{a\}. \tag{66}$$

The solution technique now proceeds as follows:

1. At the start of the first time interval, the total creep strain is zero and the increments of creep strain are taken to be zero. The thermoelastic stresses are then determined from Eqs. 65 and 66.

2. Calculate the equivalent stress from Eq. 10c.

3. Calculate the equivalent creep strain increment from both Eqs. 4 and 10c, and average.

4. Determine the incremental creep strains from Eq. 9.

5. Substitute these incremental creep strains into Eq. 65 and determine the stresses from Eq. 66.

6. Repeat steps 2 through 5 until the difference between two successive sets of strain increments is less than some prescribed value.

7. At the start of the second time interval the total creep strains are set equal to the increments of creep strain of the first time interval. The procedure of steps 1 through 6 is repeated for this and any other succeeding time intervals.

B. Creep of a Compressed Hollow Cylinder

Consider a hollow cylinder of inside radius a, outside radius b, and height 2c. The cylinder is subjected to a compressive axial force p by two rigid platens, an internal pressure p_i, and an external pressure p_0 (Fig. 2). If friction is created between platens and cylinder, the boundary conditions are

$$u = 0, \quad w = \mp Kc \qquad (z = \pm c); \tag{67}$$

$$\tau_{rz} = 0 \qquad (r = a, b);$$

$$\sigma_r = -p_i \qquad (r = a);$$

$$\sigma_r = -p_0 \qquad (r = b), \tag{68}$$

where K is a constant. The assumed trial functions for the displacements need to satisfy only the boundary conditions of Eq. 67 and the symmetry conditions

$$u = 0 \qquad (r = 0, \text{ for all } z);$$

$$w = 0 \qquad (z = 0, \text{ for all } r). \tag{69}$$

The boundary conditions of Eq. 67 and the symmetry conditions of Eq. 69 are satisfied if we assume the trial functions to have the form

$$u = b \sum_{m=2,4,\ldots}^{\infty} \sum_{n=2,4,\ldots}^{\infty} A_{mn}\left(\frac{r}{b}\right)^{m-1}\left[\left(\frac{z}{c}\right)^{n-2} - \left(\frac{z}{c}\right)^{n}\right]$$

and

$$w = -Kz + c \sum_{m=2,4,\ldots}^{\infty} \sum_{n=2,4,\ldots}^{\infty} B_{mn}\left(\frac{r}{b}\right)^{m-2}\left[\left(\frac{z}{c}\right)^{n-1} - \left(\frac{z}{c}\right)^{n+1}\right], \tag{70}$$

where A_{mn}, B_{mn}, and K are constants to be determined by the Rayleigh-Ritz method, i.e., through Eq. 65. If the material is assumed to be homogeneous and isotropic, the matrices needed for Eq. 65 are given by

$$[E] = \frac{2G}{1-2\nu}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}; \tag{71}$$

$$[B] = \begin{bmatrix} \cdots & (m-1)\left(\frac{r}{b}\right)^{m-2}\left[\left(\frac{z}{c}\right)^{n-2} - \left(\frac{z}{c}\right)^{n}\right] & \cdots\ 0\ \cdots & 0\ \cdots \\ \cdots & \left(\frac{r}{b}\right)^{m-2}\left[\left(\frac{z}{c}\right)^{n-2} - \left(\frac{z}{c}\right)^{n}\right] & \cdots\ 0\ \cdots & 0\ \cdots \\ \cdots & 0 & \cdots\ -1\ \cdots & \left(\frac{r}{b}\right)^{m-2}\left[(n-1)\left(\frac{z}{c}\right)^{n-2} - (n+1)\left(\frac{z}{c}\right)^{n}\right] \\ \cdots & 0 & \cdots\ 0\ \cdots & 0\ \cdots \\ \cdots & \left(\frac{b}{c}\right)\left(\frac{r}{b}\right)^{m-1}\left[\left(\frac{z}{c}\right)^{n-3}(n-1) - n\left(\frac{z}{c}\right)^{n-1}\right] & \cdots\ 0\ \cdots & \left(\frac{c}{b}\right)(m-2)\left(\frac{r}{b}\right)^{m-3}\left[\left(\frac{z}{c}\right)^{n-1} - \left(\frac{z}{c}\right)^{n+1}\right] \end{bmatrix}; \tag{72}$$

$$\{a\} = \begin{bmatrix} \vdots \\ A_{mn} \\ \vdots \\ K \\ \vdots \\ B_{mn} \\ \vdots \end{bmatrix}; \quad \{\eta\} = \begin{bmatrix} \epsilon_r^c + \Delta\epsilon_r^c + \alpha T \\ \epsilon_\theta^c + \Delta\epsilon_\theta^c + \alpha T \\ \epsilon_z^c + \Delta\epsilon_z^c + \alpha T \\ 0 \\ \gamma_{rz}^c + \Delta\gamma_{rz}^c \\ 0 \end{bmatrix}; \tag{73}$$

$$\{Q\} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \dfrac{-8\pi bc}{n^2 - 1}\left\{p_0 b + a\left(\dfrac{b}{a}\right)^{m-1} p_i\right\} \\ \cdot \\ \cdot \\ \cdot \\ 2PC \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{74}$$

Note that this variational approach may be extended to include anisotropic effects as well. Given an arbitrary, symmetrical temperature distribution and the desired form of creep law, the analytical outline for the variational approach is now complete.

V. CONCLUSIONS

Two methods have been presented for the analytical prediction of creep deformations in finite, hollow, right-circular cylindrical bodies under various conditions of axisymmetric mechanical and thermal loading. A generalized plane-strain approach was developed and applied both to the case of a single cylinder under radial and axial loading, and to the case of two concentric cylinders undergoing creep deformations. A second, more general variational formulation developed for the case of a single cylinder, allowed for both axial and radial variations in loading and temperature.

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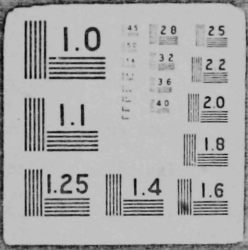

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